Chaos in the thermal regime for pinned manifolds via functional RG

Olaf Duemmer[*]

CNRS-Laboratoire de Physique Statistique, Ecole Normale Supérieure,
24 rue Lhomond, 75231 Paris Cedex 05, France

Pierre Le Doussal[†]

CNRS-Laboratoire de Physique Théorique, Ecole Normale Supérieure,
24 rue Lhomond, 75231 Paris Cedex 05, France

The statistical correlations of two copies of a d-dimensional elastic manifold embedded in slightly different frozen disorder are studied using the Functional Renormalization Group to one-loop, order $O(\epsilon = 4-d)$, accuracy. Determining the initial (short scale) growth of mutual correlations, i.e. chaos exponents, requires control of a system of coupled differential (FRG) equations (for the renormalized mutual and self disorder correlators) in a very delicate boundary layer regime. Some progress is achieved at non-zero temperature $T > 0$, where linear analysis can be used. A growth exponent a is defined from center of mass fluctuations in a quadratic potential. In the case where temperature is marginal, e.g. a periodic manifold in $d = 2$, we demonstrate analytically and numerically that $a = \epsilon(1/3 - 1/(2\ln(1/T)))$ with interesting and unexpected logarithmic corrections at low T. For short range (random bond) disorder our analysis indicates that $a = 0.083346(6)\epsilon$ with large finite size corrections.

I. INTRODUCTION

A. Overview

Systems with quenched disorder are especially sensitive to small external perturbations. This phenomenon is called chaos. Upon a perturbation of amplitude δ (an energy scale), e.g. a small change in temperature (temperature chaos) or in the disorder (disorder chaos) the configuration of the system, while only weakly affected at small scales, changes completely beyond a length scale L_δ. This overlap length diverges as $L_\delta \sim \delta^{-\alpha}$ for small δ, α being called the chaos exponent. Chaos can be studied either at $T = 0$ via the sensitivity of the ground state to perturbations, or in the thermal regime at $T > 0$. Chaos has been studied in spin glasses and other disordered systems using droplet arguments[1,2,3], numerics[4] and mean field calculations[5]. While central to disordered systems, chaos is still not fully understood. There has been some controversy as to whether the overlap length is finite or infinite (no chaos), in high dimensions as well as in mean field[5,6].

An interesting class of systems exhibiting chaos are elastic systems in random potentials such as domain walls in disordered magnets[7], or periodic systems such as charge density waves[8] and vortex lattices on disordered substrates[9,10]. They have energy-dominated glass fixed points at which temperature is irrelevant, $T_L \sim L^{-\theta}$, $\theta = d - 2 + 2\zeta$ being the free energy fluctuation exponent. The roughness exponent ζ controls the scaling of the typical deformation $u \sim L^\zeta$ with the internal size L of the pinned configurations of the elastic object. Chaos in pinned manifolds was studied mostly via scaling arguments[11]. The directed polymer ($d = 1$) was studied numerically and via analytical arguments for $N = 1$ indicating $\alpha = 1/6$ in agreement with droplets[12], and recently, on hierarchical lattices[13]. In $d = 2$ chaos was demonstrated for periodic systems near the glass transition T_g using the Cardy Ostlund RG[14].

A successful approach to disordered elastic systems is the functional renormalization group (FRG). It allows an efficient determination of the roughness of the ground state within an expansion in $\epsilon = 4-d$, to one loop[15,16,17] and recently to higher orders[18]. FRG involves a coupling constant function, $\Delta(u)$ which measures the renormalized correlator of the pinning force and becomes non-analytic beyond the (Larkin) length scale L_c where pinning produces metastability. FRG was also studied at non-zero temperature to one loop[19]. There it was shown[20] that rare thermal fluctuations (droplets) lead to the cusp of $\Delta(u)$ being rounded off inside a region of size $u \sim T$, the so-called thermal boundary layer (TBL). Recently[22] the full function $\Delta(u)$ was shown to be a proper physical observable, which describes the fluctuations of the center of mass of an elastic manifold confined by an harmonic well. It was determined by a high precision numerical calculation[23] at $T = 0$ and found to compare remarkably well with analytical predictions, already at the one loop level. The cusp in $\Delta(u)$ was also observed[23], and shown to result from so called shocks, abrupt switches of the manifold from one ground state to another, as the position of the center of the well is varied.

When applying FRG to the problem of chaos[21], one must follow not only the flow of $\Delta(u)$, but also the flow of a second FRG correlation function $D(u)$ which encodes the mutual correlations between the centers of mass of two manifold copies seeing slightly different disorders. These flows have been analyzed mostly for large scale mutual correlations. It was found[21] that the residual correlations decay to zero at large scale for (i) the 'random periodic' class (RP), i.e. a correlator $\Delta(u)$ periodic in u (which describes charge density waves and vortex lattices) and for (ii) the 'random bond' class (RB), i.e. a correlator $\Delta(u) = -R''(u)$ where $R(u)$ is a short

range function (which describes magnetic domain walls in short range disorder). Hence there is chaos with a finite L_δ. For the 'random field' class (RF), which describes magnetic interfaces in the presence of random fields, it was found that residual correlations remain non-zero. However, determining the *initial short scale* growth of mutual correlations, and hence the chaos exponents, is difficult. It requires good control of the system of coupled differential one-loop FRG equations, specifically of the separation—initially very small— between $D(u)$ and $\Delta(u)$. Since $D(u)$ remains analytic (which was confirmed numerically[23]) while $\Delta(u)$ develops a cusp at $u = 0$, one cannot use standard linear analysis. Instead, the chaos boundary layer, where the two functions differ, must be investigated: a non-trivial task in full generality.

The aim of the present paper is more modest. We study the chaos problem on the one-loop FRG equation in the thermal regime $T > 0$, where linear analysis is applicable. The easiest case to handle is when temperature does not flow under RG and there is a line of fixed points as temperature is varied. This happens for the random periodic class and $d = 2$, the case on which we focus here. Hence for each $T > 0$ there is an analytic FP, $\Delta_T(u)$, and one can use linear analysis to extract the growth exponent for mutual correlations. It is still a non-trivial task as one must perform the analysis both inside and outside the TBL, and match the two results. Surprisingly, as $\Delta_T(u)$ becomes non-analytic for $T \to 0$, one finds logarithmic corrections to the growth exponent. These corrections are confirmed by a careful numerical study of the differential equation. Despite being a special case of the full chaos problem, the random periodic class for $d = 2$ already illustrates the difficulty of obtaining the accurate behavior. An extension to the random bond class is then proposed, again within the thermal regime.

B. The model and the observables

We investigate how two identical copies of an harmonic elastic manifold embedded in frozen disorder decorrelate when they are exposed to slightly different disorder. Here we focus on interfaces, i.e. manifold whose deformations are parameterized by a real valued displacement field $u(x)$, where x is the d-dimensional internal coordinate. The system is described by the following Hamiltonian

$$H_{V,v}[u] = \sum_{i=1}^{2} \int \mathrm{d}^d x \left[\frac{1}{2}(\nabla u^i)^2 \right. \tag{1}$$
$$\left. + V_i(u^i(x), x) + \frac{1}{2}m^2(u^i(x) - v)^2 \right]$$

where the two copies $i = 1, 2$ are not mutually interacting. They are however coupled via the correlations between the two random potentials, and between the corresponding random pinning forces $F_i(u, x) = -\partial_u V_i(u, x)$,

whose correlation matrices take the form:

$$\overline{V_i(u, x)V_j(u', x')} = R_{ij}^{(0)}(u - u')\delta^d(x - x') \tag{2}$$

$$\overline{F_i(u, x)F_j(u', x')} = \Delta_{ij}^{(0)}(u - u')\delta^d(x - x'). \tag{3}$$

with $\Delta_{ij}^{(0)}(u) = -R_{ij}^{(0)''}(u)$. We use the superscript (0) to denote bare disorder, to distinguish it from the renormalized disorder defined below. The form of these correlation functions differentiates the three main universality classes: (i) random periodic (RP): a periodic $R^{(0)}(u)$ (ii) random bond (RB) a short range function $R^{(0)}(u)$ (iii) random field (RF) $R^{(0)}(u) \sim \sigma|u|$ and $\Delta^{(0)}(u)$ short range. One way of realizing eq. (2) is to consider two disorder copies of the following form: $V_i(u, x) = V(u, x) \pm \delta W(u, x)$ with V and W mutually uncorrelated Gaussian disorders of zero mean, and δ a very small parameter. In the simplest case V and W are in the same class with identical correlator. Note that a quadratic well, i.e. a mass term, has been added to confine each elastic manifold copy to a mean position v, another external parameter. Its role is discussed below. One is usually interested in the limit $m \to 0$.

The statistical correlations of the two ground states are measured by the standard observables, i.e. the correlation functions:

$$C_{ij}(x - x') = \overline{\langle(u^i(x) - u^i(x'))\rangle\langle(u^j(x) - u^j(x'))\rangle} \tag{4}$$

where $\overline{..}$ denotes disorder averages. We will also study non-zero, though low, temperature, denoting the thermal averages by $\langle..\rangle$ (using the canonical partition function $Z_{V,v} = \int Du\,e^{-H_{V,v}[u]/T}$). For a single copy one expects mean square deformations due to disorder to result in $C_{11}(x) \sim c_{11}|x|^{2\zeta}$ at large $L_c \ll x \ll \frac{1}{m}$, ζ being the roughness exponent, independent of temperature at low T (and in the whole glass phase where the manifold is pinned near its ground state with only a few active thermal excitations). Standard arguments that assume the existence of a single diverging scale, the overlap length $L_\delta \sim \delta^{-1/\alpha}$, suggest the following scaling form for the two-point correlation function between different copies at large $L_c \ll x \ll \frac{1}{m}$:

$$C_{12}(x) = x^{2\zeta}\Phi(\delta x^\alpha). \tag{5}$$

The overlap length separates correlated from uncorrelated scales, and depends sensitively on the small difference δ between the bare disorders of the two copies. In analogy with chaotic dynamical systems, in which tiny differences in initial conditions are amplified via the Lyapunov exponent(s) to large scale differences, one introduces the *chaos* exponent α. This exponent is a measure of how the two copies effectively split as scale increases. Qualitatively, this splitting is characterized by a dimensionless scale-dependent parameter which grows under RG as $\tilde{\delta}_l = \delta e^{\alpha l}$ where l is the log-scale, e.g. $l = \ln L$. This behavior is suggested by droplet arguments, first developed for spin glasses where they predict[2] $\alpha = d_f/2 - \theta$

with $-\theta$ being the thermal eigenvalue and d_f the fractal dimension of the droplets. In the case of manifolds the same formula was proposed[3,11] for the SR disorder class, with $d_f = d$, namely $\alpha = d/2 - \theta = (\epsilon - 4\zeta)/2$.

Another observable, introduced recently[22], quantifies the fluctuations of the center of mass confined by an harmonic well. One defines $u^i(x;v) = \langle u^i(x) \rangle$ the thermally averaged position. It depends on the position of the center of the harmonic well v. One denotes the center of mass of the manifold by $\bar{u}^i(v) = L^{-d} \int d^d x u^i(x;v)$, L^d being the system volume. The second cumulant of its position, as the disorder is varied:

$$m^4 \overline{(\bar{u}^i(v) - v)(\bar{u}^j(v') - v')} = L^{-d}\Delta_{ij}(v - v') \quad (6)$$

defines the renormalized pinning force cross-correlator $\Delta_{ij}(u)$. At zero temperature these functions measure the correlations between the shocks in the two copies. These abrupt jumps of the manifold as the center v is varied do not occur exactly at the same place in the two copies, which results in the cross correlator $\Delta_{12}(u)$ remaining a smooth function of u. This was confirmed in[23] where simultaneous shocks in the two copies were examined and $\Delta_{12}(u)$ was computed numerically. Another useful quantity is the free energy of each copy $\hat{V}_i(v) = -T \ln Z_{V,v}$: for each disorder configuration it is a random function of the well center position v, hence one defines the renormalized second cumulant of the potential:

$$\overline{\hat{V}_i(v)\hat{V}_j(v')} = L^d R_{ij}(v - v') \quad (7)$$

and it is easy to see that $\Delta_{ij}(u) = -R_{ij}''(u)$.

C.   Functional RG approach

The functional RG method allows us to compute from first principles the observables defined above, namely the correlation functions of eq. (4) and the renormalized correlators of eq. (6) and (7). FRG is based on the replicated field theory, and proceeds via a loop expansion for which — at zero temperature — the small parameter is $\epsilon = 4 - d$. Here we give only results, and refer to[22,24] for reviews on the method. The FRG flow of the renormalized correlators R_{ij} were derived in[21]. The FRG flow equations for the correlators R_{ij} and Δ_{ij}, defined in eq. (6) and (7), are found by computing the effective action and its flow equation, as the mass is varied. One defines $m \equiv m_l = m_0 e^{-l}$ where l is the usual RG logarithmic scale[29]. We define the rescaled dimensionless force correlator $\tilde{\Delta}_{ij}$ via $\Delta_{ij}(u) = A_d^{-1} m_l^{\epsilon - 2\zeta} \tilde{\Delta}_{ij}(um_l^\zeta)$, with $A_4^{-1} = 8\pi^2$, and the roughness exponent ζ reflecting the self-affine scaling property of the manifold. A rescaled temperature is also defined as $T_l = T m_l^\theta$. We simplify notation by writing $\Delta \equiv \tilde{\Delta}_{11}$ for the one-copy correlator and $D \equiv \tilde{\Delta}_{12}$ for the two-copy correlator. Their flow equations are[21], respectively, to one loop:

$$\partial_l \Delta(u) = (\epsilon - 2\zeta)\Delta(u) + \zeta u \Delta'(u) - \Delta'(u)^2 \quad (8a)$$
$$+ [\Delta(0) - \Delta(u)]\Delta''(u) + T_l \Delta''(u)$$
$$\partial_l D(u) = (\epsilon - 2\zeta)D(u) + \zeta u D'(u) - D'(u)^2 \quad (8b)$$
$$+ [\Delta(0) - D(u)]D''(u) + T_l D''(u)$$

The zero temperature FRG equation are obtained by setting $T_l = 0$.

Note that the temperature is irrelevant (for $\theta > 0$) because $T_l = T e^{-\theta l}$ (setting $m_0 = 1$). However, the $T_l \Delta''(u)$ term keeps the correlation functions smooth for any non-zero T. If one studies the FRG to only one loop, one can also use the (more qualitative) Wilson RG procedure, which consists of varying the short scale momentum cutoff $\Lambda_l = \Lambda_0 e^{-l}$. In that case the mass cutoff is unnecessary: one can set $m = 0$ and estimate the correlation functions of eq. (4) at non-zero momentum.

It is important to note that the one-copy correlator in eq. (8) evolves independently, whereas the two-copy correlator is linked to the former via its value at the origin $\Delta(0)$. This small but crucial difference entails opposing behaviors for the two correlators[25]: the one-copy correlator converges to its stable fix point, whereas the two-copy correlator diverges towards another fixed point. The difference between the two correlators is denoted

$$\Theta_l \equiv \Delta(0) - D(0) \quad (9)$$
$$\Theta_{l=0} \; \delta^d(x - x') =$$
$$\frac{1}{2} \overline{(F_1(0,x) - F_2(0,x))(F_1(0,x') - F_2(0,x'))}$$

One easily sees from eq. (8) that Θ_l generates an additional term $\Theta_l D''(u)$ in the flow of $D(u)$, compared to that of $\Delta(u)$. Hence one can think of Θ_l as an effective temperature (its bare value before renormalization is $\Theta_{l=0} \sim \delta^2$) but its flow under RG is very different from that of the real temperature T. In fact, its flow is determined self-consistently by the two equations. While in general it smoothes the form of $D(u)$ which hence remains analytic[25], it is often relevant, i.e. grows with l, by contrast to temperature.

To compute the observables of eq. (6) and (7) as a function of m one must solve the above flow equations. The difference (eq. (9)) is a direct measure of the fluctuations of the distance between the (thermally averaged) centers of mass $\bar{u}^i(v)$ of the two copies:

$$\overline{(\bar{u}^1(v) - \bar{u}^2(v))^2} = A_d L^{-d} m_l^{-d-2\zeta} \Theta_l \quad (10)$$

The correlation function of eq. (4) is more delicate to compute. The general formula $C_{ij}(q) = \Delta_{ij,l}(0)/(q^2 + m^2)^2$ is exact for $q = 0$, $m = m_l$, and holds to $O(\epsilon)$ (i.e. to one loop accuracy) for $q \sim m$. It also holds for $q = \Lambda_l$, $m = 0$ within the (one loop) Wilson scheme and hence provides an estimate for the correlation function at large scale (and for $q \gg m$):

$$C_{ij}(q) \approx A_d q^{-4} [\tilde{\Delta}_{l,ij}(0) e^{(2\zeta - \epsilon)l}]_{l=\ln(\frac{\Lambda_0}{q})} \quad (11)$$

However caution is required when using this estimate, even when computing the simpler, small q behavior. For instance, for the random periodic (RP) universality class (for which $\zeta = 0$) the estimate is correct only for $d > 2$. In general one must examine more carefully the FRG for the non-local part of the effective action[26]: in $d = 2$, $\theta = 0$ the two gradient term becomes dominant and yields an extra $\ln(1/q)$ in the single copy correlation and the famous $\ln^2|x|$ in the real space correlation[30]. The study needed to elucidate the initial growth regime being even more subtle, our results here will mostly concern the $q = 0$, center-of-mass behavior of eq. (10) and (6).

Finally note that the thermal correlations:

$$C_{ij}^{th}(x - x') = \tag{12}$$
$$\overline{\langle (u^i(x) - u^i(x'))(u^j(x) - u^j(x')) \rangle}$$
$$- \overline{\langle (u^i(x) - u^i(x')) \rangle \langle (u^j(x) - u^j(x')) \rangle}$$

always vanish identically for $i \neq j$, because the two copies do not interact[31].

The outline of the paper is as follows. We start by studying the simplest case of the random periodic class. In section II we focus on zero temperature, review known results and explain why the problem is difficult. In section III we study the RP problem at $T > 0$ in the so called marginal case of $\theta = 0$ (i.e. in $d = 2$) where temperature does not flow. This allows to use linear analysis. In section IV analytical predictions and numerical analysis are compared. In section V we generalize these results to the random bond class, and RP for $d = 4 - \epsilon$. The results are summarized and discussed in the conclusion.

II. RANDOM PERIODIC CLASS: ZERO TEMPERATURE CONSIDERATIONS

A. Flow of single copy and fixed point of the random periodic problem (CDW, Bragg glass)

Here we study the random periodic class (RP), which has logarithmic roughness, i.e $\zeta = 0$. The single-copy correlator is a periodic function $\Delta(u + 1) = \Delta(u)$ of normalized period one, and obeys the following flow equation at $T = 0$, from eq. (8):

$$\partial_l \Delta(u) = \epsilon \Delta(u) - \Delta'(u)^2 \tag{13}$$
$$+ (\Delta(0) - \Delta(u))\Delta''(u)$$

where $\epsilon = 4 - d$. This flow is well understood[15,17,19]. Beyond the Larkin scale — here $m = m_c = m_0 e^{l_c}$ — it develops a non-analyticity (cusp) at $u = 0$, and flows towards an attractive fixed point:

$$\Delta^*(u) = \frac{\epsilon}{36}(1 - 6u(1 - u)) \quad , \quad u \in [0,1] \tag{14}$$

Note that this correlator presents a cusp at $u = 0$ because of the periodicity condition $\Delta(u + 1) = \Delta(u)$.

B. Flow of the two-copy correlator

The flow of the two-copy correlator is more intricate, due to the (scale dependent) coupling Θ_l to the single-copy correlator.

$$\partial_l D(u) = \epsilon D(u) - D'(u)^2 \tag{15}$$
$$+ (\Theta_l + D(0) - D(u))D''(u)$$
$$= \epsilon D(u) - D'(u)^2$$
$$+ (\Delta(0) - D(u))D''(u)$$

The only difference between the two equations is the term $\Theta_l D''(u)$. If one starts with a very small $\Theta_0 \sim \delta^2 \ll 1$, the two correlators $\Delta(u)$ and $D(u)$ remain practically identical up to the Larkin scale l_c, and the difference $\Theta_l = \Delta(0) - D(0)$ remains small. One finds that it grows as $\Theta_l \sim e^{2\tilde{a}_L l}\Theta_0$, with $\tilde{a}_L = (\epsilon - 2\zeta)/2$, since in most of this regime one can neglect the non-linearities. Near the Larkin scale, non-linearities become important and Δ'' becomes large as the cusp develops. Once Θ_l grows such that $\Theta_l D''(0) \sim \Theta_l \Delta''(0) \sim \epsilon D(0)$, which occurs very near l_c, the two-copy correlator $D(u)$ starts to differ from the one-copy correlator. By analogy with temperature, one expects this difference to be mostly confined to a boundary layer (BL) of width $u \sim \Theta_l \ll 1$ around $u = 0$. This BL is called chaos BL to distinguish it from the thermal BL $u \sim T_l$ (absent at zero temperature). While the one-copy correlator flows towards its fixed point and develops a cusp (eq. (14)), the two-copy correlator remains analytic.

The flow beyond the Larkin scale is non-trivial. In Ref.[21] the final behavior of the flow for $l \to \infty$ was examined. It was found that ultimately $D(u)$ flows to $D(u) = 0$ for the RP class, hence there are no residual correlations between the two copies. Here we address a different question. We are interested in the first phase of the FRG flow, i.e. we study how the difference Θ_l grows beyond the Larkin scale.

Clearly Δ^* is a fixed point both for $D(u)$ and $\Delta(u)$. However while Δ flows towards its attractive fixed point Δ^*, D is repelled by it. Let us assume that the one-copy correlator has already reached its fixed point $\Delta(u) = \Delta^*(u)$ (eq. (14)). We then have to solve the FRG equations for the flow of the two-copy correlator D, from which we can deduce the behavior of the difference $\Theta_l = \Delta^*(0) - D(0)$:

$$\partial_l D(u) = \epsilon D(u) - D'(u)^2$$
$$+ (\Delta^*(0) - D(u))D''(u) \tag{16}$$
$$\partial_l \Theta_l = -\partial_l D(0)$$

One can always write:

$$D(u) = \Delta^*(u) + f(u, l) \tag{17}$$

where, during the initial growth phase, $f(u, l)$ remains small (in a sense to be made precise below). The problem we face at $T = 0$, is that the one-copy fixed point

correlator is non-analytic, whereas the two-copy correlator is analytic, with the cusp rounded off inside the chaos boundary layer. Hence the function $f(u,l)$ should be equally non-analytic, to cancel the fixed point cusp and leave a smooth analytic function $D(u)$.

An important property of $f(u,l)$ arises from the potentiality constraint. $\Delta_{ij}(u) = -R''_{ij}(u)$ can at most have a cusp singularity[32] and for the RP class the R_{ij} are periodic, while for the RB class they must be short ranged, which implies:

$$\int du f(u,l) = 0 \qquad (18)$$

for both the RP and the RB class (the integration domains being $u \in [0,1]$ and $u \in [0,\infty]$, respectively.)

We now attempt a linear expansion around the fixed point. We need eigenfunctions $f(u,l)$ that are non-analytic and that obey the zero-mean constraint of eq. (18). Although linear analysis is not necessarily valid inside the boundary layer, outside (for $u \gg \Theta_l$) it is appropriate.

C. Eigenvalue problem of the $T = 0$ linearized flow equation

One starts from:

$$\Delta^*(0) - D(u) = \Delta^*(0) - \Delta^*(u) + f(u,l) \qquad (19)$$

$$= \frac{\epsilon}{6} u(1-u) + f(u,l) \qquad (20)$$

and inserts it into the flow equation (16) keeping linear terms only. We assume the eigenvector flows as:

$$f(u,l) = \exp(2al) f_a(u) \qquad (21)$$

which provides one definition of a growth exponent a, as discussed below. One gets:

$$0 = (\epsilon - 2a)f(u) + \frac{\epsilon}{6}[u(1-u)f(u)]'' \qquad (22)$$

$$= 4(1 - \frac{3a}{\epsilon})f(u)$$
$$+ 2(1-2u)f'(u) + u(1-u)f''(u)$$

For any value of a this linear second order differential equation has two types of solutions on the interval $u \in [0,1]$, even and odd about $u = 1/2$. We must select the even one (since the correlators are symmetric: $\Delta(-u) = \Delta(u)$ and $D(-u) = D(u)$ combined with periodicity). The even solution for general eigenvalue a can be expressed in terms of the hypergeometric function $_2F_1$

$$f_a(u) = {}_2F_1\left(\frac{3}{4} - \frac{1}{4}\sqrt{25 - 48\frac{a}{\epsilon}}, \qquad (23) \right.$$
$$\left. \frac{3}{4} + \frac{1}{4}\sqrt{25 - 48\frac{a}{\epsilon}}; \frac{1}{2}; (1-2u)^2 \right)$$

where $_2F_1(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!}$, with $(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)$, being Gauss' hypergeometric series. It provides a convergent series for $0 < u < 1$, whose behavior near $u = 0$ is:

$$f_a(u) = A_a\left[\frac{1}{u} + B_a \ln u (1 + b_1(a)u + b_2(a)u^2 + ..) \right]$$
$$+ c_0(a) + c_1(a)u + c_2(a)u^2 + .. \qquad (24)$$

with $B_a = 12\frac{a}{\epsilon} - 6$ and

$$A_a = \frac{\sqrt{\pi}}{4\Gamma[\frac{3}{4} - \frac{1}{4}\sqrt{25 - 48\frac{a}{\epsilon}}]\Gamma[\frac{3}{4} + \frac{1}{4}\sqrt{25 - 48\frac{a}{\epsilon}}]} \qquad (25)$$

Hence for generic a there is a non-integrable divergence at $u = 0$ together with terms non-analytic in $|u|$. Note a particularly simple solution for $a = \epsilon/2$: $f_a(u) = 1/(u(1-u))$. It is however not integrable at $u = 0$.

For $\int_0^1 f_a(u)$ to be defined (i.e. finite) one needs $A_a = 0$, which gives:

$$\frac{3}{4} \pm \frac{1}{4}\sqrt{25 - 48\frac{a}{\epsilon}} = -n \qquad (26)$$

where n is a positive integer. It yields a series of values:

$$a = a_n = \left(\frac{1}{3} - \frac{1}{3}n^2 - \frac{1}{2}n \right)\epsilon \quad , \quad n = 0, 1, 2, .. \qquad (27)$$

for which the hypergeometric series becomes a polynomial of finite order. For the highest value:

$$a = \frac{\epsilon}{3} \quad , \quad f_a(u) = c_0 \qquad (28)$$

the eigenfunction is a constant. The next one is $a = -\frac{\epsilon}{2}$ and corresponds to $f_a(u) = c_e(1 - 5u(1-u))$. A more detailed analysis is performed in appendix A. It is found that none of these eigenfunctions satisfy the potentiality condition of eq. (18) $\int_0^1 f_a(u) = 0$.

This dilemma of non-zero mean eigenfunctions will be overcome in the next section, by studying the FRG flow at non-zero temperature. Setting $T > 0$ induces the cusp to round off within a boundary layer around $u = 0$, and permits solutions of the eigenvalue problem that have zero mean value.

More generally, the above analysis is valid outside the boundary layer (BL), be it a thermal BL $u \sim T_l$ or the chaos BL $u \sim \Theta_l$. One expects the blow-up of the eigenfunction near the origin to be rounded off within the BL. We now examine how.

III. RANDOM PERIODIC UNIVERSALITY CLASS: NON-ZERO TEMPERATURE $T > 0$ IN THE MARGINAL CASE ($\theta = 0$, $d = 2$).

To escape from the difficulty of a non-analytic fixed point Δ^* we now consider the problem at non-zero temperature. In this section we focus on the simplest case,

$\theta = d - 2 = 0$, where temperature does not flow under RG. Hence there is a line of analytic fixed points Δ_T^*, indexed by $T > 0$, which converge to $\Delta^* = \Delta_{T=0}^*$ at the end of the line $T \to 0^+$. Around each of these fixed points linear analysis is then possible for all u. The physical temperature T introduces a cutoff scale for u that allows us to find a coherent solution to the eigenvalue problem. Of course here $\epsilon = 2$, hence the one-loop results are expected to be approximate. Several recent works have found that the one-loop scheme provides reasonable approximations of exponents and a clear, qualitatively correct picture for this model[23,26,27].

A. One-copy correlator at non-zero temperature T

Temperature enters the FRG flow equation for the one-copy correlator in a natural way:

$$\partial_l \Delta_T(u) = \epsilon \Delta_T(u) - \Delta_T'(u)^2 \qquad (29)$$
$$+ [T + \Delta_T(0) - \Delta_T(u)]\, \Delta_T''(u).$$

The resulting fixed point equation is integrable:

$$\epsilon \Delta_T^*(u) = \Delta_T^{*\prime}(u)^2 \qquad (30)$$
$$- (T + \Delta_T^*(0) - \Delta_T^*(u))\Delta_T^{*\prime\prime}(u)$$
$$= \frac{1}{2}[(\Delta_T^*(u) - \Delta_T^*(0) - T)^2]''$$

and is implicitly solved by quadrature[19]:

$$u = \sqrt{\frac{3}{2\epsilon}}\, G(T, T + \Delta_T^*(0) - \Delta_T^*(u)) \qquad (31)$$

with

$$G(a,b) \equiv \int_a^b \frac{\mathrm{d}y\, y}{\sqrt{(y - T)(y - y_-)(y_+ - y)}} \qquad (32)$$

$$4y_\pm = 3\Delta_T^*(0) + T \qquad (33)$$
$$\pm \sqrt{3(3\Delta_T^*(0) + T)(\Delta_T^*(0) + 3T)}\,,$$
$$\text{with} \quad y_- < 0 < T < y_+.$$

The constraint $\frac{1}{2} = \sqrt{\frac{3}{2\epsilon}}G(T, y_+)$ yields the value of the fixed point correlator at zero $\Delta_T^*(0)$ as a function of T.

The finite temperature correlator fixed point $\Delta_T^*(u)$, given implicitly by equation (31) reduces to the non-analytic zero-temperature correlator of eq. (14) as $T \to 0$. Notice how the finite temperature T rounds off the cusp within a boundary layer of width $\sim T$: The curvature at the origin becomes finite $\Delta_T^{*\prime\prime}(0) = -\frac{\epsilon \Delta_T^*(0)}{T}$, and within the boundary layer, for $u \ll T$, the following series expansion holds:

$$\Delta_T^*(u) = \Delta_T^*(0) - \frac{\Delta_T^*(0)T}{2}\left(\frac{u}{T}\right)^2 \qquad (34)$$
$$\times \left[1 - \frac{3\Delta_T^*(0) + T}{12}\left(\frac{u}{T}\right)^2\right.$$
$$\left.\times \left[1 + \frac{15\Delta_T^*(0) + T}{30}\left(\frac{u}{T}\right)^2\right]\right] + O\left(\left(\frac{u}{T}\right)^8\right)$$

These are the first terms of a systematic low temperature expansion[19]:

$$\Delta_T^*(u) = \Delta_T^*(0) - \sum_{k \geq 1} T^k \phi_k(u/T) \qquad (35)$$

with $\phi_k(0) = 0$ valid inside the TBL, i.e. for $u/T = O(1)$. The following matching conditions hold at large $x = u/T$:

$$\phi_k(x) \sim c_k|x|^k \qquad (36)$$
$$\sum_{k \geq 1} c_k|u|^k = \Delta_{T=0}^*(0) - \Delta_{T=0}^*(u) \qquad (37)$$

since for $u = O(1)$, $\Delta_T^*(u) = \Delta_{T=0}^*(u) + O(T)$. The first scaling form is $\phi_1(x) = \phi(x) - 1$ where we define:

$$\phi(x) = \sqrt{1 + \left(\frac{\epsilon x}{6}\right)^2}, \qquad (38)$$

as can be verified by inserting the expansion of eq. (35) into the FRG flow eq. (29) and collecting orders in $O(T)$. The low temperature expansion will be detailed and generalized in section V, where ϕ_2 will also be computed. For the present purpose the following low temperature form of the BL is sufficient:

$$\Delta_T^*(0) - \Delta_T^*(u) = \qquad (39)$$
$$T\left(\sqrt{1 + \left(\frac{\epsilon u}{6T}\right)^2} - 1\right) + T^2 \phi_2(u/T) + O(T^3)$$

It reproduces well the $T \to 0$ limit $\frac{\epsilon u}{6}$, as well as the first term of the power series expansion around $u = 0$ of eq. (34).

B. Linearization of non-zero T flow equation

The FRG flow equation at $T > 0$ for the two-copy correlator, assuming the one-copy correlator has reached its fixed point $\Delta_T^*(u)$, reads:

$$\partial_l D(u) = \epsilon D(u) - D'(u)^2 \qquad (40)$$
$$+ (T + \Delta_T^*(0) - D(u))D''(u)$$

with $\partial_l \Theta_l = -\partial_l D(0)$. Since $D(u) = \Delta_T^*(u)$ is now a (analytic) fixed point of this equation, we define:

$$D(u) = \Delta^*(u) + f(u, l) \qquad (41)$$

and perform a linear analysis for small f, i.e. we write $f(u, l) = e^{2al}f_a(u)$ and look for eigenfunctions. This can be done at any T, at least numerically, using the implicit form of the exact fixed point given in the last section. At low temperature it can be done analytically, provided one distinguishes the two regimes, $u \sim T$ (TBL) and $u \sim 1$. In the second regime the analysis becomes identical, to leading order in T, to the one performed directly at zero temperature in section II C. In the TBL, it is natural to look for solutions of the form $f_a(u) = \tilde{f}(u/T)$. The matching between the two regimes will be studied in the

next section. Inserting in the linearized version of eq. (40) and (41) and using the low T expansion of eq. (35) we obtain:

$$0 = (\epsilon - 2a)\tilde{f}(x) \tag{42}$$
$$+ \frac{\epsilon^2}{T}\frac{\mathrm{d}^2}{\mathrm{d}x^2}[(\phi(x) + T\phi_2(x))\tilde{f}(x)]$$

from which a systematic low T expansion of $\tilde{f}(x)$ can be obtained.

For $a = \epsilon/3$, one finds that a simple ansatz almost solves it (in an approximate sense given below) namely:

$$\tilde{f}(x) = \frac{1}{\phi(x)} - \frac{12T}{\epsilon}\ln(\phi(x)). \tag{43}$$

noting that, from eq. (38):

$$\phi'(x) = \frac{x}{36\phi} = \frac{\sqrt{\phi^2 - 1}}{6\phi} \tag{44a}$$

$$\phi''(x) = \frac{1}{36\phi}\left(1 - \frac{x\phi'}{\phi}\right) = \frac{1}{36\phi^3} \tag{44b}$$

inserting into the right hand side of the above equation leads to:

$$0 = (1 - 2/3)(1/\phi - \frac{12T}{\epsilon}\ln\phi) \tag{45}$$
$$+ \frac{\epsilon}{T}[1 - \frac{12T}{\epsilon}\phi\ln\phi + \frac{\phi_2}{\phi}]''$$
$$= \frac{1}{3\phi} - 12\left[\phi''(1 + \ln\phi) + \frac{\phi'^2}{\phi}\right] + \epsilon\left(\frac{\phi_2}{\phi}\right)'' + O(T)$$
$$= \frac{1}{3\phi} - \frac{1}{3\phi} + \frac{\ln\phi}{3\phi^3} + \epsilon\left(\frac{\phi_2}{\phi}\right)'' + O(T)$$

Hence the dominant terms cancel, and the remaining term $\frac{\ln\phi}{3\phi^3}$ is found to be subdominant at large u going as $\sim \frac{\ln u}{u^3}$ as $u \to \infty$. A more complete analysis is done in section V A, where it is shown that the term $\epsilon(\phi_2/\phi)''$ is also subdominant. For now suffice it to note that the form of eq. (43) is exact to dominant order in T for all x, and to next order in T it reproduces the exact large x behavior. Thus, surprisingly, we find evidence for a *logarithmic* correction to the eigenfunction, emerging at non-zero temperature $T > 0$.

C. Logarithmic temperature dependence of the eigenvalue

Let us examine the correction, induced by temperature, to the eigenvalue. Assume that the main contribution to the eigenvalue correction comes, to first order, from the much larger regime outside the boundary layer where the eigenfunction is given by the expression of eq. (23). Assume further that a is close to the value $a = \epsilon/3$ and expand the expression for the corresponding

eigenfunction in powers of $\delta a = \frac{a}{\epsilon} - \frac{1}{3}$, as well as around $u = 0$, using eq. (24):

$$f_a(u) = 1 + \delta a\left(\frac{1}{u} - 3 - 2\ln(4u)\right) \tag{46}$$
$$+ O(\delta a^2, u^2, \delta a\,u) \quad, \quad u = O(1)$$

The divergence at $u = 0$ is rounded off inside the TBL. The previous paragraph gives the expression inside the TBL, to leading order in T:

$$f_a(u) = \frac{\delta a\,\epsilon}{6T}\frac{1}{\sqrt{1 + \left(\frac{\epsilon u}{6T}\right)^2}} \quad, \quad u = O(T) \tag{47}$$

where we have multiplied with a constant in order to match the $1/u$ term of eq. (46) for large u/T.

The eigenfunction is now integrable and we enforce the condition of zero mean, expressed as $\int_0^{1/2}\mathrm{d}u f_a(u) = 0$ when taking into account that the eigenfunction is symmetric about $u = 1/2$. We split the integral into two parts, inside $u < kT$ and outside $u > kT$ the boundary layer (any large constant k will do), and we find up to order $O(T, \delta a)$:

$$0 = \int_0^{\frac{1}{2}}\mathrm{d}u f_a(u) \tag{48}$$
$$= \frac{\delta a\,\epsilon}{6T}\int_0^{kT}\frac{\mathrm{d}u}{\sqrt{1 + \left(\frac{\epsilon u}{6T}\right)^2}}$$
$$+ \int_{kT}^{\frac{1}{2}}\mathrm{d}u\left(1 + \delta a\left(\frac{1}{u} - 3 - 2\ln 4u + O(u)\right)\right)$$
$$0 = \frac{1}{2} - \delta a\left(\ln T + O(1)\right) + O(T).$$

This shows that for the eigenfunction to integrate to zero, i.e. for the $\ln T$ divergence to be compensated, the eigenvalue acquires a logarithmic temperature dependence.

$$\delta a = -\frac{1}{2\ln\frac{1}{T}} + O(\delta a^2) \tag{49}$$

In fact, inserting this expression for the eigenvalue correction back into the eigenfunction expansion of eq. (46) and normalizing by $6T/\delta a\epsilon$, we retrieve exactly the asymptotic form of the eigenfunction inside the boundary layer (eq. (43)).

$$\frac{6T}{\delta a\epsilon}f_a(u) = \frac{6T}{\epsilon u} - \frac{12T}{\epsilon}\ln\frac{u}{T} + \dots \tag{50}$$
$$= \lim_{\frac{u}{T}\to\infty}\left(\frac{1}{\sqrt{1 + \left(\frac{\epsilon u}{6T}\right)^2}} - \frac{12T}{\epsilon}\ln\sqrt{1 + \left(\frac{\epsilon u}{6T}\right)^2}\right)$$

including all logarithmic terms. Hence our solution satisfies the required conditions, zero mean and matching between inside and outside the TBL. Note that it was necessary to not only include the dominant contribution

in the TBL, but also the subdominant one to logarithmic accuracy.

In principle the low T expansion can be pursued to higher orders. The second order $O(\delta a^2)$ is much harder to calculate though, because it requires us to exactly integrate the non-divergent part of $f_a(u)$ over the whole interval $[T, 1/2]$.

IV. NUMERICAL STUDY OF THE RANDOM PERIODIC CLASS AT $T > 0$ ($\theta = 0$, $d = 2$)

To test the subtle mechanism for the selection of the eigenvalue based on the boundary layer matching, we now turn to a numerical analysis.

A. Shooting to solve eigenvalue problem

We numerically solve the FRG flow equation for the two-copy correlator (eq. (40)) linearized around the exact implicit solution of the one-copy correlator at finite T (eq. (31)). The eigenvalue equation is

$$0 = (\epsilon - 2a - \Delta_T^{*\prime\prime}(u))f_a(u) - 2\Delta_T^{*\prime}(u)f_a'(u)$$
$$+ (T + \Delta_T^*(0) - \Delta_T^*(u))f_a''(u) \qquad (51)$$

with periodic boundary conditions: $f_a(0) = f_a(1) = 1$ and $f_a'(0) = f_a'(1) = 0$. Again, since the eigenfunction is even about $u = 1/2$, it suffices to consider the interval $u \in [0, 1/2]$, requiring $f_a'(1/2) = 0$. We set ϵ to unity since it plays no role.

1. Numerical details

We need a continuous numerical representation for the fixed point correlator $\Delta_T^*(u)$, whose analytical properties are given in section III A. The first step is to evaluate $\Delta_T^*(0)$ for given temperature T, from the constraint $1/2 = \sqrt{3/2}\, G(T, y_+)$. We have to find $\Delta_T^*(0)$ such that the elliptic integral is exactly one half. This is easily done using the Brent method. The integrand has a divergence $\sim 1/\sqrt{u}$ at each limit, which is handled by a change of variables to $v^2 = u$. We use a Romberg integration routine to efficiently get the desired precision.

We then use the implicit expression of eq. (31) to calculate a discrete representation $\{u_i(\Delta_i), \Delta_i; i = 1..N\}$ of the correlator. The Δ_i are chosen such that we have a sufficiently fine discretization inside the boundary layer $u \sim T$, even for very small values of T, and are chosen less dense outside the boundary layer, to reduce the total number N of support points.

A cubic spline interpolation of the discrete function $\Delta_i(u_i)$ allows us to obtain a continuous representation of the correlator $\Delta_T^*(u)$. At the origin $u = 0$ we take advantage of the exact series expansion of the correlator

of eq. (34), matching the series expansion to the spline at about $u \sim 0.01T$.

With this continuous numerical representation of the correlator fixed point, we can solve the eigenvalue problem of eq. (51) to arbitrary precision, using the shooting method: make an initial guess for the eigenvalue a, and integrate the ordinary differential equation starting from the initial condition $f_a(0) = 1, f_a'(0) = 0$, by means of a standard integration routine (e.g. odeint). Aiming for periodic boundary conditions $f_a'(1/2) = 0$, one finds the eigenvalue $a(T)$.

In this way, we calculate the eigenvalues $a(T)$ and eigenfunctions $f_a(u)$ as a function of the temperature T to arbitrary precision. We need however to go to very small values of $T \simeq 10^{-16}$, to clearly see the logarithmic dependence on T of the eigenvalue (eq. (49)). This requires quadruple precision for the numerics.

B. Numerical results

Indeed, we find numerically that the largest physical eigenvalue $\frac{a}{\epsilon}$ is equal to one third plus corrections logarithmic in the cutoff T (see figure 1, upper half). This confirms our analytical finding of logarithmic corrections to the eigenvalue. There exists one larger eigenvalue $\frac{a}{\epsilon} = 1/2$. It is easy to see that the exact eigenfunction for this eigenvalue is:

$$f_{a=\epsilon/2}(u) = \frac{K}{\Delta_T^*(0) - \Delta_T^*(u) + T} \qquad (52)$$

for all u inside and outside the TBL (which correctly matches the eigenfunction $f_{a=\epsilon/2} \sim 1/(u(1-u))$ for $u \gg T$). This eigenvalue does not acquire any corrections in the cutoff T, and most importantly, its corresponding eigenfunctions are strictly positive, they do not have any zeros. This means they cannot have zero mean and hence do not correspond to a correlator of the RP class.

We recall the first analytical terms of the eigenfunction of eq. (50, with $\epsilon = 1$):

$$f_T(u) = \frac{1}{\sqrt{1 + \frac{u^2}{36T^2}}} - 12T\,\ln\sqrt{1 + \frac{u^2}{36T^2}} \quad (53)$$

$$\lim_{u/T \to \infty} f_T(u) = \frac{6T}{u} - 12T\,\ln\frac{u}{T}$$

This expression compares rather well to the numerically calculated solution (figure 3). Moreover, setting equation (53) to zero gives us the first order term of the zero of the eigenfunction $u_0 = \frac{1}{2\ln 1/T}$. The prefactor $\frac{1}{2}$ is exactly the one found in the numerical data of the first zero of the eigenfunction (see figure 1, lower half). Thus the numerical results confirm our analytical analysis, providing us with a coherent picture of the solution of the
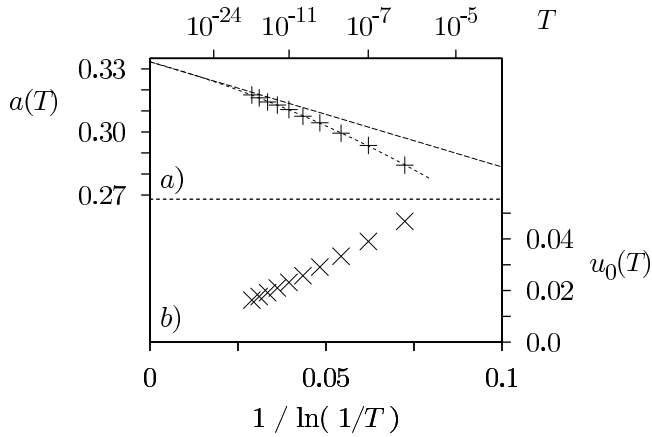
FIG. 1: *Slow logarithmic dependence on cutoff T.* $a)(+)$ Eigenvalue a and $b)(\times)$ zero u_0 of the first eigenfunction approach their zero temperature values $a)1/3$ $b)0$ as $1/\ln[1/T]$ (we set $\epsilon = 1$). The linear coefficient of the fitted numerical eigenvalue data coincides with the theoretical value of $-1/2$. Equally, the linear coefficient of the fitted zero of the eigenfunction is identical to the theoretical value of $1/2$. The ordinate scale is linear in $1/\ln[1/T]$. As a guide, the corresponding value for T is given on the upper ordinate scale.

linearized FRG flow equation at non-zero T. This is represented schematically in Fig. 2.

As the cutoff T approaches zero, the eigenfunction shifts more and more weight into the ever smaller BL, in order to still fulfill the zero-mean constraint $\int f = 0$. While this picture is satisfactory for any fixed T it is still not clear whether it could help to solve the problem directly at $T=0$. The question of how in the limit $T \to 0$ this eigenfunction develops the non-analyticity necessary to balance the cusp in the one-copy correlator is especially subtle given that there are two regimes with different T-scaling properties. The boundary layer disappears as $u_{\rm BL} \sim T$, whereas the zero of the eigenvector approaches zero as $u_0 \sim 1/\ln[1/T]$ (figure 1, lower half).

C. Consequence of logarithmic correction

The unusual logarithmic correction to the eigenvalue — caused by a finite cutoff length — implies that it is very hard to calculate the latter by means of intuitive numerical approaches. If for example one simply tries to numerically integrate the FRG flow equation, one necessarily introduces a cutoff length ($\sim N^{-1}$, N being the number of discretization intervals in real space, or the highest frequency mode in Fourier space). This cutoff length has exactly the same effect as the finite temperature cutoff, i.e. preventing access to smaller lengthscales. Thus even at zero real temperature, any finite numerical
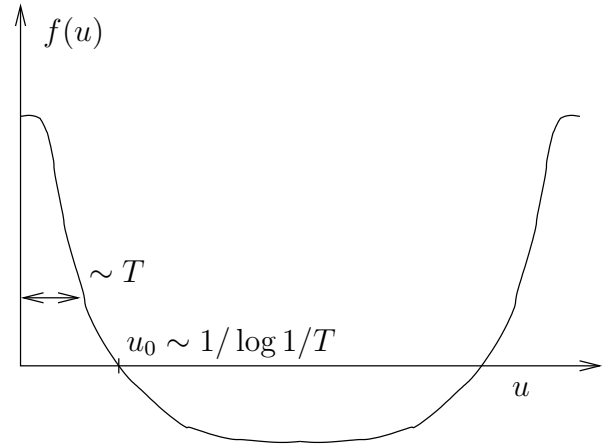

FIG. 2: *Different regimes of eigenfunction.* The eigenfunction has three different regimes, separated by two crossover values, represented schematically here. Near zero, the eigenfunction presents a boundary layer of width $u \sim T$ as found in Fig. 3. Outside the boundary layer, the eigenfunction falls off, and crosses zero at u_0. This zero has a logarithmic dependence on temperature, introducing a second crossover, which scales differently from the width of the boundary layer $u_0 \sim 1/\log 1/T$. For even larger values of u, the eigenfunction flattens but remains negative, such that the overall integral of the RP eigenfunction is strictly zero.
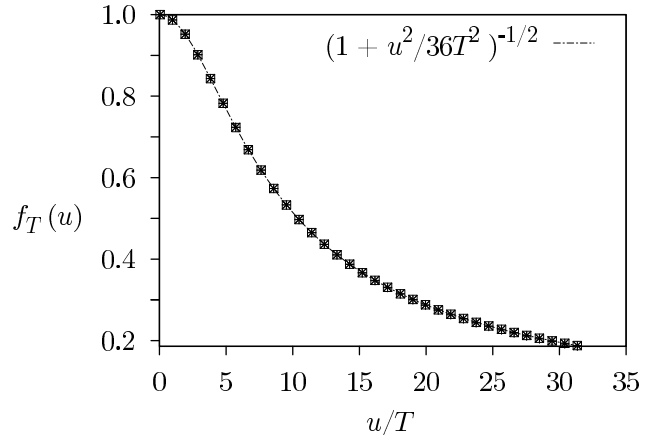

FIG. 3: *Scaling form of eigenfunctions.* Inside a boundary layer of width $u \sim T$, the eigenfunctions present a functional form independent of cutoff T. Numerical data for various temperatures $T = 10^{-4}...10^{-8}$ is indistinguishable from the theoretical expression $f_T(u) = 1/\sqrt{1+u^2/36T^2}$.

cutoff introduces an immediate and non-negligible correction to the eigenvalue of order $\ln[N]^{-1}$. For example, a reasonably large $N \approx 10^6$ leads to a correction of order 10% to the eigenvalue.

A logarithmic correction to an eigenvalue, not unsimilar to the present situation, has been noticed in the context of a propagating wave front[28].

V. GENERAL ANALYSIS AT $T > 0$ AND EXTENSION TO OTHER CLASSES

We turn from the simple case ($d = 2$, RP class) where the temperature does not flow, to the general problem at non-zero temperature. The two equations (8) are studied with the temperature allowed to flow, i.e. $\theta > 0$. Despite $T_l = Te^{-\theta l}$ flowing to zero, it is still possible to use linear analysis, as we will show. We look again for a zero mean eigenfunction, placing us in the RP class for $d > 2$, and the RB class.

A. TBL for one copy correlator

We start by solving more accurately the first equation (8) and write:

$$
\begin{align}
\Delta(u) &= \Delta^*(u) \quad , \quad u = O(1) \tag{54a}\\
\Delta(u) &= \Delta(0) - T_l\phi_1(\chi u/T_l) - T_l^2\phi_2(\chi u/T_l) \tag{54b}\\
&\quad + O(T_l^3) \quad , \quad u = O(T_l) \tag{54c}\\
\Delta(0) &= \Delta^*(0) - T_l\gamma_1 + O(T_l^2) \tag{54d}
\end{align}
$$

with $T_l = Te^{-\theta l}$. One must have $\phi_n(0) = 0$, and $\phi_1(x) \sim |x|/6$ at large x to fit the cusp, hence the choice $\chi = 6|\Delta^{*\prime}(0^+)|$. The zero temperature FP $\Delta^*(u)$ can now be of RP type, as well as RB (in which case ζ is non-zero and determined by the FP equation and the SR boundary condition). From the analysis performed in Appendix B one finds:

$$
\phi_1(x) = \sqrt{1 + x^2/36} - 1 = \phi(x) - 1 \tag{55}
$$

and $\chi^2 = \epsilon^2\tilde{\chi}^2 = 36\Delta'(0^+)^2 = 36(\epsilon - 2\zeta)\Delta^*(0)$ ($\tilde{\chi} = 1$ for the periodic FP) using $\phi_1''(0) = 1/36$. From which we recover the zero temperature fixed point (eq.14) in the large argument limit $[\phi_1(x \to \infty) = \frac{|x|}{6} + \frac{3}{|x|} + \dots]$. To next order one finds:

$$
\begin{align}
\phi_2(x) &= \frac{1}{\chi^2\phi(x)}[12(6 - 2\epsilon + 2\zeta)(\phi(x) - 1) \tag{56}\\
&\quad + x^2(1 - \gamma_1) + \frac{1}{6}x^2\phi(x)(\zeta - \epsilon))\\
&\quad - 3(4 + \zeta - \epsilon)\, x\, \text{ArcSinh}(\frac{x}{6}))]
\end{align}
$$

Note that $\phi_2(x) \sim (\zeta - \epsilon)x^2/(6\chi^2) + O(x)$ at large x. This is compatible with $\Delta^{*\prime\prime}(0^+) = (\epsilon - \zeta)/3$. Note also that:

$$
\begin{align}
\lim_{x\to\infty}\left(\frac{\phi_2(x)}{\phi(x)}\right)'' &= \frac{72}{\chi^2 x^3}\Big(15\theta + 7\epsilon - 22\zeta \tag{57}\\
&\quad + 3(\epsilon + 2\theta - 3\zeta)\ln(\frac{3}{x})\Big) + O(x^{-4})
\end{align}
$$

as promised in section III B, hence validating the approximate solution given there.

B. Equation for the two copy correlator

Now we define the solution of the second equation in (8) to be:

$$
D(u) = \Delta(u) + f(u) \tag{58}
$$

and we study the resulting equation for $f(u)$ in a linear approximation:

$$
\begin{align}
\partial_l f(u) &= (\epsilon - 2\zeta)f(u) + \zeta u f'(u) \tag{59}\\
&\quad + \frac{d^2}{du^2}[(\ T_l + \Delta(0) - \Delta(u)\ )f(u)]
\end{align}
$$

The only neglected term is $-\frac{1}{2}\frac{d^2}{du^2}f(u)^2$ on the r.h.s. One can write:

$$
\begin{align}
\partial_l f(u) &= (\epsilon - 2\zeta)\ f(u) + \zeta u f'(u) \tag{60}\\
&\quad + \frac{d^2}{du^2}[(\ T_l\phi(\chi u/T_l) + T_l^2\phi_2(\chi u/T_l)\\
&\quad\quad + O(T_l^3)\ )f(u)] \quad , \quad u \sim T_l\\
\partial_l f(u) &= (\epsilon - 2\zeta)\ f(u) + \zeta u f'(u) \tag{61}\\
&\quad + \frac{d^2}{du^2}[(\ \Delta^*(0) - \Delta^*(u)\ )f(u)]\\
&\quad + O(T_l) \quad\quad\quad\quad\quad , \quad u \sim O(1)
\end{align}
$$

where we have explicitly separated the inside from the outside of the TBL.

1. Eigenfunction inside the TBL

Inside the TBL we look for a solution of the form:

$$
f(u) = \frac{\chi}{6T_l}\tilde{f}(x = \chi u/T_l) \quad , \quad u \sim T_l \tag{62}
$$

As we will see below, the prefactor $1/T_l$ is crucial for obtaining a correct matching to the $u = O(1)$ regime.

Equation (61) gives:

$$
\begin{align}
\partial_l\tilde{f}(x) &= (\epsilon - 2\zeta - \theta)\tilde{f}(x) + (\zeta - \theta)x\tilde{f}'(x) \tag{63}\\
&\quad + \frac{\chi^2}{\tilde{T}_l}\frac{d^2}{dx^2}[(\ \phi(x) + T_l\phi_2(x) + O(T_l^2)\ )\tilde{f}(x)]
\end{align}
$$

The solution seems to admit the expansion:

$$
\tilde{f}(x) = e^{2al}\left(\frac{1}{\phi(x)} + T_l\psi(x) + O(T_l^2)\right) \tag{64}
$$

where:

$$
\begin{align}
0 &= (\epsilon - 2\zeta - 2a - \theta)\frac{1}{\phi(x)} - (\zeta - \theta)x\frac{\phi'(x)}{\phi(x)^2} \tag{65}\\
&\quad + \chi^2\frac{d^2}{dx^2}\left[\frac{\phi_2(x)}{\phi(x)} + \phi(x)\psi(x)\right]
\end{align}
$$

This yields:

$$
\begin{align}
\psi(x) &= \frac{\alpha}{\phi(x)} - \frac{\phi_2(x)}{\phi(x)^2} + \frac{1}{\chi^2}\Big(36(\epsilon + \theta - 4\zeta - 2a) \tag{66}\\
&\quad + (2a - \epsilon + 3\zeta)\frac{6x\text{ArcSinh}(\frac{x}{6})}{\phi(x)}\Big)
\end{align}
$$

with α undetermined. At large x one has:

$$\psi(x) = \frac{36}{\chi^2}(2a - \epsilon + 3\zeta)\ln\left(\frac{x}{3}\right) \tag{67}$$
$$+ \frac{36}{\chi^2}(\epsilon + \theta - 4\zeta - 2a) - \frac{6}{\chi^2}(\zeta - \epsilon) + O(1/x)$$

If one defines the "eigenfunction" $f_a(u)$ through $f(u) = e^{2al}f_a(u)$, it has the following behavior at large u/T_l:

$$f_a(u) \sim \frac{\chi}{6T_l}\left[\frac{6T_l}{\chi u} + \frac{36}{\chi^2}(2a - \epsilon + 3\zeta)T_l \ln\left(\frac{u}{3T_l}\right)\right] \tag{68}$$

consistent, up to a normalization with the result (50) for the RP class in $d = 2$, setting $\zeta = 0$, $\chi = \epsilon$, $a = 1/3$. This expression will be matched to the small u behavior of the eigenfunction $f_a(u)$ in the regime $u = O(1)$, which contains a small $1/u$ term. Note however that $f_a(u) \sim (\chi/(6T_l))/\sqrt{1 + \chi^2 u^2/(36T_l^2)}$ in the TBL, hence $f_a(0)$ has a different dependence in l that $f_a(u)$ for $u = O(1)$, i.e. it is a non-uniform eigenvector.

2. eigenfunction outside the TBL

For u of order unity one must study the linear differential equation (62) containing the zero temperature fixed point $\Delta^*(u)$.

Let us start with the RP class (for $d > 2$). Since the fixed point $\Delta^*(u)$ does not change form (apart from the overall factor of ϵ already taken into account) we expect the same behavior as for $d = 2$. From the ansatz of eq. (62) one sees that for $\zeta = 0$ the behavior (eq. (68)) identifies with eq. (50) which matches the one for $u = O(1)$, (eq. (46)), as discussed already in section III C. Hence for the RP class[33], we find the same growth exponent $a = \epsilon/3$ for the function outside the TBL (for $u = O(1)$). But from the discussion of the end of the previous paragraph, the growth exponent of:

$$\Theta_l \sim e^{2\tilde{a}l} \tag{69}$$

is determined by $f(0)$ inside the TBL, hence $\tilde{a} = a + \frac{\theta}{2}$. This non-uniformity is the main difference with the case $d = 2$.

For the RB class the equation for the eigenvector for $u = O(1)$ reads:

$$0 = (\epsilon - 2\zeta - 2a)f(u) + \zeta u f'(u) \tag{70}$$
$$+ \frac{d^2}{du^2}\left[(\Delta^*(0) - \Delta^*(u))f(u)\right]$$

This equation involves the RB fixed point, which is non trivial[15] and was determined with high accuracy in[18] together with the value for $\zeta = 0.208298063$. An analysis near $u = 0$ shows that any solution of eq. (70) has the form of eq. (24) at small u, with $B_a = (\epsilon - 3\zeta - 2a)/\Delta^{*\prime}(0^+)$. We will assume that one can then proceed as for the RP case and look for the value $a = a_{\mathrm{RB}}$

such that $A_{a_{\mathrm{RB}}} = 0$. This is equivalent to the shooting problem of solving eq. (70) imposing that $f(0) = 1$ and that $f(u)$ decay at infinity. This fixes a unique and non-trivial value for a_{RB}. We have solved this shooting problem numerically using Mathematica[34], and found $a_{\mathrm{RB}} = 0.083346(6)\epsilon$. The corresponding eigenvector is everywhere positive and integration of eq. (70) easily leads to the following constraint:

$$(\epsilon - 3\zeta - 2a)\int_{0^+}^{\infty}du f(u) = -\Delta'(0^+)f(0). \tag{71}$$

Indeed one finds $a_{\mathrm{RB}} < (\epsilon - 3\zeta)/2$ (the 0^+ means that the domain excludes the TBL). Once this eigenfunction is determined in the region $u = O(1)$ the method to satisfy the zero integral condition over the full axis including the TBL is the same as for the RP class. First, one checks that the solution (eq. (68)) in the TBL matches correctly the small u behavior (eq. (24)) of the $u = O(1)$ regime, using $B_a = (\epsilon - 3\zeta - 2a)/\Delta^{*\prime}(0^+)$. Second, one has again $A_{a_{\mathrm{RB}}+\delta a} \sim \delta a$ and one can proceed as in section III C. One finds that the zero mean condition again leads to logarithmic corrections $a_l = a_{\mathrm{RB}} - K/ln(1/T_l) = a_{\mathrm{RB}} - K/(\theta l)$ where $K = \int_{0^+}^{\infty}du f(u)/\partial_a A_a|_{a=a_{\mathrm{RB}}}$ given by eq. (71). The main difference with the RP class, besides the value of the growth exponent, is that the logarithmic temperature corrections are also (weakly) l-dependent[35].

VI. SUMMARY AND DISCUSSION

In this paper we have studied the problem of two manifolds pinned in slightly different random potentials at the same temperature. We have written the coupled FRG equations for the single and two-copy correlators, $\Delta(u)$ and $D(u)$, and temperature, to one loop accuracy. We have investigated how the difference between the two copies increase with scale. We have focused on zero-momentum $(q = 0)$ quantities and specifically we have computed the fluctuations of the difference in (thermally averaged) center-of-mass positions $\bar{u}^i(v)$, $i = 1, 2$, of the two copies, in the presence of a uniform confining harmonic potential (of curvature $m^2 = m_0^2 e^{-2l}$) centered at a common position $u = v$. This observable is exactly given by:

$$\frac{1}{2}\overline{(\bar{u}^1(v) - \bar{u}^2(0) - v)^2} \tag{72}$$
$$= A_d(Lm)^{-d}m^{-2\zeta}(\Delta(vm^{\zeta}) - D(vm^{\zeta}))$$

and can be seen to measure the r.m.s shift in position of $(L/L_m)^d$ roughly independent pieces of manifolds of typical size $L_m \sim 1/m$ (hence the factor $(Lm)^{-d}$ from the central limit theorem). The deviation of the one-copy center of mass from the center of the well is typically $\bar{u}^i(v) - v \sim O(m^{-\zeta})$. We have defined the two growth exponents:

$$\Theta_l := \Delta(0) - D(0) = \tilde{C}e^{2\tilde{a}l} \tag{73}$$
$$\Delta(u) - D(u)|_{u=O(1)} = Ce^{2al}f_a(u) \tag{74}$$

allowing for the possibility that $\tilde{a} \neq a$. The coefficients C and $\tilde{C}$ vanish when the difference in disorder δ between the two copies is taken to zero.

We have obtained these exponents at non-zero temperature, using linear analysis and matching the regime $u = O(1)$ to the regime $u \sim T_l = Te^{-\theta l}$ in eq. (74), corresponding respectively to $v = O(m^{-\zeta})$ and $v = O(Tm^{\theta-\zeta})$, the width of shocks, in eq. (72). For the random periodic class for $d \geq 2$ and for the random bond class we found, with $\epsilon = 4 - d$:

$$\tilde{a} = a + \frac{\theta}{2} \tag{75}$$

$$a_{\mathrm{RP}} = \left(\frac{1}{3} - \frac{1}{2\ln(1/T_l)} \right) \epsilon \tag{76}$$

$$a_{\mathrm{RB}} = \left(0.083346(6) - \frac{K}{\ln(1/T_l)} \right) \epsilon \tag{77}$$

The result for the $d = 2$, RP class has been thoroughly checked via a numerical simulation. The other results assume that the scenario demonstrated in that case can be extended, which appears to be consistent. To confirm it further would require extensive numerics or a more complete analytical study.

Since it originates from linear analysis, the growth of eq. (73) is valid only for a limited range of scales $l = \ln(m/m_0)$ (i.e. of masses m in eq. (72)). From eq. (59), where the term $(f^2)''$ has been neglected the linear analysis is valid only, in the regime $u = O(1)$, as long as $Ce^{2al} \ll 1$. Beyond that scale the growth is non linear. However, since the eigenvector is peaked in the TBL the condition of validity of our analysis is more stringent. One can write the condition $(f^2)'' \ll T_l f''$ and substitute $f \sim (C/u)e^{2al}$ with $u \sim T_l$. This yields $Ce^{2al}/T_l \sim \tilde{C}e^{2\tilde{a}l} \ll T_l$ which, not surprisingly, can be written as:

$$\Theta_l \ll T_l \tag{78}$$

i.e. the width of the chaos BL (which exists at $T = 0$) is smaller than the TBL. This is the condition for validity of the thermal regime and linear analysis. Qualitatively, it means that the thermal width of the shocks should be larger than the typical shift in their relative position in the two copies.

More issues remain to be understood. If one defines the overlap length $L_\delta \sim e^{l_\delta}$ from $\Theta_{l_\delta} \sim 1$ one can argue that the chaos exponent is $\alpha = \tilde{a}$ in the thermal regime. Note however that this assumes that $\tilde{C} \sim \delta^2$, a natural condition, but which may be spoiled at scales around the Larkin length where we do not have good control on the flow. Concerning the definition of L_δ from eq. (5), using eq. (11) as was discussed there, we need a more precise calculation of the scale dependence of the non-local terms in the FRG. Finally, the present analysis does not solve the question of the zero temperature chaos boundary layer when the condition of eq. (78) is violated, although it gives some insight into selection mechanisms for growth eigenvalues in such FRG equations.

To conclude, we have made a step towards solving the intricate non-linear coupled FRG flow of the force correlators. We have found an interesting result for the growth exponent of the elastic manifold in frozen disorder when the problem can be solved using a linearized flow equation. We have shown that it acquires a surprisingly large logarithmic correction when a finite cutoff length is present (here the temperature). More work is necessary to completely solve the problem and to establish the exact relation between the eigenvalue found here and the chaos exponent.

APPENDIX A: ANALYSIS OF LINEARIZED EQUATION

We give a detailed account of the linearized flow equation at zero temperature (eq. (22)). As we are looking for a continuous solution periodic on $u \in [0,1]$, we are only interested in the even solutions. The first solution that springs to mind is $f(u) = 1/u(1-u)$, corresponding to the eigenvalue $a = \epsilon/2$. It is however not integrable at $u = 0$.

In order to get an idea of possible integrable eigenfunctions, we look for finite series solutions. Applying the Frobenius method, we postulate a power-series solution of the form

$$f_a(u) = u^r \sum_{n=0}^{\infty} c_n u^n \tag{A1}$$

insert it into equation (22) and equate coefficients of each term in the power series. The lowest power (indicial equation) provides the two possible values for $r = 0, -1$.

In the case of $r = 0$, the recurrence relation for the coefficients is:

$$c_{m+1} = c_m \frac{m(m+3) - 4 - 12a/\epsilon}{(m+1)(m+2)} \tag{A2}$$

Notice that the ratio of coefficients tends to one $\lim_{m\to\infty} \frac{c_{m+1}}{c_m} = 1$, hence the infinite series does not converge uniformly on the interval $[0..1]$. We thus have to demand that the series terminate after a finite number of terms, i.e. that the mth coefficient be zero $c_{m+1} = 0$, giving us the eigenvalues $\tilde{a}_m$:

$$\tilde{a}_m = \frac{\epsilon}{3} \left(1 - \frac{m(m+3)}{4} \right) \tag{A3}$$

Here we include even and odd solutions, whereas the eigenvalues a_n of section II C are $a_n = \tilde{a}_{m=2n}$. The case $r = -1$ does not lead to an independent solution, but points us into the right direction. Noting that the derivative of $\ln \frac{1-u}{u}$ is given by $-1/u(1-u)$, a term that cancels the polynomial occurring under the second derivative of equation (22), we insert the following ansatz

$$f_a(u) = \frac{q(u)}{u(1-u)} + p(u) \ln \frac{1-u}{u} \tag{A4}$$

with $p(u), q(u)$ polynomials in u. Equating terms in $\ln\frac{1-u}{u}$ as well as terms in $1/u(1-u)$ leads to the following two equations:

$$0 = 4(1 - 3a/\epsilon)p(u) + 2(1 - 2u)p'(u) \quad (A5a)$$
$$+ u(1-u)p''(u),$$
$$0 = 6(1 - 2a/\epsilon)q(u) + u(1-u)q''(u) \quad (A5b)$$
$$- (1-2u)p(u) - 2u(1-u)p'(u).$$

$p(u)$ obeys the same equation as $f_a(u)$, hence the second type of logarithmic solutions has the same set of eigenvalues as the first type (eq. (A3)), and these eigenvalues (including $a = \epsilon/2$) are the only ones possible.

$p(u)$ gives us $q(u)$, again by comparing coefficients, and thus we have the complete set of finite-series solutions. The first few are listed in table I. None of the finite-series eigenfunctions, however, fulfill the zero-mean condition $\int_0^1 du f_a(u) = 0$. From equation (22) one has

$$\int_0^1 du f_a(u) = -\frac{1}{6(1 - 2a/\epsilon)}\int_0^1 du[u(1-u)f_a(u)]''. \quad (A6)$$

In the case of the first type of solutions

$$\int_0^1 du f_a(u) = \frac{1}{6(1 - 2a/\epsilon)}[f_a(0) + f_a(1)], \quad (A7)$$

which is non-zero for the even solutions we are interested in.

In the case of the logarithmic solutions (eq. (A4)):

$$\int_0^1 du f_a(u) = -\frac{1}{6(1 - 2a/\epsilon)}\int_0^1 du\left[q(u)\right. \quad (A8)$$
$$\left. + u(1-u)\log\frac{1-u}{u}p(u)\right]''$$
$$= \frac{1}{6(1-2a/\epsilon)}\left[q'(0) - q'(1) + p(1) - p(0)\right], \quad (A9)$$

which is not zero either, for the even solutions.

APPENDIX B: THERMAL BOUNDARY LAYER

We now insert the ansatz of eq. (54a) into the FRG equations, for $\partial_l(\Delta(u) - \Delta(0))$ and $\partial_l\Delta(0)$ respectively, collecting orders in $O(T_l)$, and using $\partial_l \equiv -\theta T_l \partial_{T_l}$ one finds to first order:

$$0 = \phi_1''(0) - \phi_1''(x) - \frac{1}{2}(\phi_1(x)^2)'' \quad (B1)$$
$$0 = (\epsilon - 2\zeta)\Delta^*(0) - \chi^2\phi_1''(0) = 0 \quad (B2)$$

The solution is given in the text. The second order in $\tilde{T}_l$ yields:

$$(\theta - \zeta)x\phi_1'(x) - (\theta + \epsilon - 2\zeta)\phi_1(x) = \quad (B3)$$
$$\chi^2((\phi_1(x)\phi_2(x))'' + \phi_2''(x) - \phi_2''(0))$$
$$-(\theta + \epsilon - 2\zeta)\gamma_1 = \chi^2\phi_2''(0) \quad (B4)$$

whose solution is:

$$\phi_2(x) = \frac{1}{\phi(x)}\left[\frac{1}{2}\phi_2''(0)x^2\right. \quad (B5)$$
$$+ \frac{1}{\chi^2}\left((12\epsilon - 48\zeta + 36\theta)(\phi(x) - 1)\right.$$
$$+ \frac{1}{2}x^2(\theta + \epsilon - 2\zeta) + \frac{1}{6}x^2\phi(x)(\zeta - \epsilon)$$
$$\left.\left. - 3(\epsilon - 3\zeta + 2\theta)\, x\, \mathrm{ArcSinh}(\frac{x}{6})\right)\right]$$

Fixing the free parameter $\phi_2''(0)$ by means of eq. (B4), the above simplifies into:

$$\phi_2(x) = \frac{1}{\chi^2\phi(x)}\left[12(\epsilon - 4\zeta + 3\theta)(\phi(x) - 1)\right. \quad (B6)$$
$$+ \frac{1}{2}x^2(\theta + \epsilon - 2\zeta)(1 - \gamma_1) + \frac{1}{6}x^2\phi(x)(\zeta - \epsilon)$$
$$\left. - 3(\epsilon - 3\zeta + 2\theta)\, x\, \mathrm{ArcSinh}(\frac{x}{6})\right]$$

This can be further simplified using $\theta = 2 - \epsilon + 2\zeta$ to yield the formula given in the text. One can check that this agrees with the result of[19] once corrected for a misprint ($f_3(x) \rightarrow f_3(x) + x^2(\epsilon - \zeta)/3$ in the formula given in Appendix D3 of[19]). There the TBL expansion of the quantity $y(u) = (\Delta(u) - \Delta(0) - T)^2 = T^2 f_2(\chi u/6) + \chi^{-2}T^3 f_3(\chi u/6) + ..$ was computed.

m	$\tilde{a}_m/\epsilon$	$f_m^{even}(u)$	$f_m^{odd}(u)$
-1	$\frac{1}{2}$	$\frac{c_e}{u(1-u)}$	$\frac{c_o u}{u(1-u)}$
0	$\frac{1}{3}$	c_e	$c_o\left(\frac{1-2u}{u(1-u)} + 2\ln\frac{1-u}{u}\right)$
1	0	$c_e\left(\frac{1-12u(1-u)}{u(1-u)} + 6(1-2u)\ln\frac{1-u}{u}\right)$	$c_o(1-2u)$
2	$-\frac{1}{2}$	$c_e(1 - 5u(1-u))$	$c_o\left(\frac{(1-2u)(1-30u(1-u))}{u(1-u)} - 12(1 - 5u(1-u))\ln\frac{1-u}{u}\right)$

TABLE I: First few eigenfunctions of the linearized flow equation (22) at zero temperature.

[*] Electronic address: duemmer@lps.ens.fr

[†] Electronic address: ledou@lpt.ens.fr

[1] S.R. McKay et al., Phys. Rev. Lett. **48** 767 (1982).

[2] A. J. Bray and M. A. Moore, Phys. Rev. Lett. **58** 57 (1987).

[3] D.S. Fisher, D.A. Huse, Phys. Rev. B **38** 373 (1988), D. S. Fisher and D. A. Huse, Phys. Rev. B **43** 10728 (1991).

[4] J. Kisker et al., Phys. Rev. B **53** 6418 (1996), J. Kisker, H. Rieger, Phys. Rev., B **58**, R8873 (1998).

[5] T. Rizzo and A. Crisanti, Phys. Rev. Lett. **90** 137201 (2003), I. Kondor, A. Vegso, J. Phys. A **26** L641 (1993).

[6] M. Ney-Nifle, cond-mat/9707172. A. Billoire, E. Marinari, cond-mat/0202473.

[7] S. Lemerle et al., Phys. Rev. Lett. **80** (1998) 849.

[8] G. Grüner, Rev. Mod. Phys. **60**, 1129 (1988).

[9] G. Blatter et al., Rev. Mod. Phys. **66** 1125 (1994); T. Nattermann and S. Scheidl, Adv. Phys. **49** 607 (2000).

[10] T. Giamarchi and P. Le Doussal in *Spin glasses and random fields* Ed. A. P. Young, World Scientific, Singapore, 1998.

[11] Y. Shapir, Phys. Rev. Lett. **66** 1473 (1991).

[12] Y. Zhang, Phys. Rev. Lett. **59** 2125 (1987) M. V. Feigel'man and V. M. Vinokur, Phys. Rev. Lett. **61** 1139 (1988). M. Sales and H. Yoshino, Phys. Rev. E **65**, 066131 (2002).

[13] R. A. Silveira, J.P. Bouchaud, Phys. Rev. Lett. **93**, 015901 (2004)

[14] T. Hwa, D.S. Fisher, Phys. Rev. Lett. **72** 2466 (1994).

[15] D. S. Fisher, Phys. Rev. Lett. **56**, 1964 (1986)

[16] T. Nattermann et al., J. Phys. (Paris) **2**, 1483 (1992). O. Narayan, D. S. Fisher, Phys. Rev. B **46**, 11520 (1992).

[17] T. Giamarchi and P. Le Doussal, Phys Rev. Lett. **72** 1530 (1994) and Phys. Rev. B **52** 1242 (1995).

[18] P. Chauve, P. Le Doussal and K.J. Wiese, Phys. Rev. Lett. **86** (2001) 1785. P. Le Doussal, KJ. Wiese and P. Chauve, Phys Rev B **66**, 174201 (2002) and Phys Rev E **69**, 026112 (2004).

[19] P. Chauve, T. Giamarchi, and P. Le Doussal, *Phys. Rev. B* **62**, 6241 (2000)

[20] L. Balents and P. Le Doussal, Europhys. Lett. **65**, 685 (2004), Annals of Phys. **315** 213 (2005).

[21] P. Le Doussal, cond-mat/0505679, *Phys. Rev. Lett.* **96**, 235702 (2006)

[22] P. Le Doussal, cond-mat/0605490, Europhys. Lett. **76** No 3 (November 2006) 457-463.

[23] A. Middleton, P. Le Doussal, K. Wiese, cond-mat/0606160, Phys. Rev. Lett. **98** (2007) 155701.

[24] P. Le Doussal, K.J. Wiese and P. Chauve, Phys. Rev. E **69** 026112 (2004); K.J. Wiese and P. Le Doussal, cond-mat/0611346, see also lecture notes at Windsor School of CMP at http://www.lancs.ac.uk/users/esqn/windsor04/ handouts/ledoussal.pdf

[25] This is the usual situation but there can be exceptions such as the case of subdominant chaos, where the perturbation is of shorter range than the original disorder, e.g. $V_i = V + \delta W$ where V is RF and W is RB.

[26] G. Schehr and P. Le Doussal, cond-mat/0607657, Phys. Rev. B **75** (2007) 184401.

[27] G. Schehr and P. Le Doussal, cond-mat/0501199, Europhys. Lett., **71** (2), pp. 290-296 (2005) and *Phys. Rev. Lett.* **80**, 849 (1998).

[28] E. Brunet and B. Derrida, Computer Physics Comm. **121** 376 (1999), Phys. Rev. E **57** 2597 (1997). E. Brunet, B. Derrida, A. H. Mueller, and S. Munier, *Phys. Rev. E* **73**, 056126 (2006)

[29] Strictly speaking, for the observable of eq. (6) one has the initial condition of the flow $\Delta(u) = \Delta^0(u)$ for $m = +\infty$, i.e. $l = -\infty$.

[30] As was shown there, such a term can be generated at $T = 0$ only from the non-analyticity in the disorder correlator. Since the two copy disorder correlator R_{12} remains analytic, such a term should not be generated in the two copy mutual correlation C_{12} which hence should remain a single logarithm. This is consistent with numerical results (G. Schehr, H. Rieger unpublished).

[31] This agrees with a droplet estimate in the case of two almost degenerate wells in each copy, each thermally occupied with probabilities p_i and $1 - p_i$ respectively, provided the probabilities of joint thermal occupations are products of the form $p_1 p_2$. It is obvious that in a given disorder sample (V_1, V_2) thermal occupations are statistically independent in each copy if there are no interactions between the copies. However, there are obviously correlations between the random variables p_1 and p_2 with respect to the measure on the disorder.

[32] That is $R'(0^{\pm}) = 0$, i.e. no supercusp.

[33] Note again for $\zeta = 0$ the exact eigenfunction of (70) $f_{a=\epsilon/2}(u) = \frac{K}{\Delta^*(0)-\Delta^*(u)}$ which does not satisfy the required conditions.

[34] We thank Kay Wiese for providing high precision approximations for Δ^*.

[35] This dependence being much weaker than the leading exponential one, it implies an additional l dependence of the eigenvector which is subdominant and neglected here.